LITERATURE CITED

1. A. V. Suvorov and G. D. Rabinovich, "Transitional process in a thermodiffusion apparatus with transverse streams," Inzh.-Fiz. Zh., 39, No. 1, 86-96 (1980).
2. G. D. Rabinovich, Fractionation of Isotopes and Other Mixtures by Thermodiffusion [in Russian], Atomizdat, Moscow (1981).

SOLUTION OF ONE PROBLEM OF HEAT CONDUCTION IN A REGION WITH A MOVING BOUNDARY BY THE METHOD OF EXPANSION IN ORTHOGONAL WATSON OPERATORS

N. N. Kuznetsova

UDC 536.24

The problem of heat conduction with branching of the heat flux at moving boundaries is solved by the method of expansion in orthogonal Watson operators.

The problem of the temperature distribution along two linear heat conductors with thermally insulated lateral surfaces is considered. We assume that a linear combination of the unknown functions and their derivatives is assigned at the moving ends of these heat conductors, and the ends of a heat conductor move by a linear law. This problem comes down to the solution of the system [1]

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{a^2}\frac{\partial u}{\partial t},\quad \frac{\partial^2 v}{\partial y^2}=\frac{1}{b^2}\frac{\partial v}{\partial t} \tag{1}$$

in the region $\ell_1 + pt \leq x \leq \ell_2 + pt$, $\ell_2 - \ell_1 > 0$; $\ell_3 + qt \leq y \leq \ell_4 + qt$, $\ell_4 - \ell_3 > 0$, $-\infty < t < +\infty$, with the following initial (at $t = -\infty$) and boundary conditions at the boundaries moving by the linear law:

$$u|_{t=-\infty}=0,\quad v|_{t=-\infty}=0, \tag{2}$$

$$\left(\alpha_{11}\frac{\partial u}{\partial x}+\alpha_{12}u\right)\Big|_{x=l_1+pt}+\left(\alpha_{13}\frac{\partial v}{\partial y}+\alpha_{14}v\right)\Big|_{y=l_3+qt}=h_1(t), \tag{3}$$

$$\left(\alpha_{21}\frac{\partial u}{\partial x}+\alpha_{22}u\right)\Big|_{x=l_1+pt}+\left(\alpha_{23}\frac{\partial v}{\partial y}+\alpha_{24}v\right)\Big|_{y=l_4+qt}=h_2(t), \tag{4}$$

$$\left(\alpha_{31}\frac{\partial u}{\partial x}+\alpha_{32}u\right)\Big|_{x=l_2+pt}+\left(\alpha_{33}\frac{\partial v}{\partial y}+\alpha_{34}v\right)\Big|_{y=l_3+qt}=h_3(t), \tag{5}$$

$$\left(\alpha_{41}\frac{\partial u}{\partial x}+\alpha_{42}u\right)\Big|_{x=l_2+pt}+\left(\alpha_{43}\frac{\partial v}{\partial y}+\alpha_{44}v\right)\Big|_{y=l_4+qt}=h_4(t), \tag{6}$$

where p, q, ℓ_k, α_{jk} (i, k = 1, 2, 3, 4) are assigned positive constants; $\alpha_{11}^2 + \alpha_{21}^2 \neq 0$; $\alpha_{32}^2 + \alpha_{42}^2 \neq 0$; $\alpha_{13}^2 + \alpha_{33}^2 \neq 0$; $\alpha^2_{24} + \alpha_{44}^2 \neq 0$; $h_k(t)$ (k = 1, 2, 3, 4) are assigned functions of time satisfying the condition

$$h_k(t)\exp\left(-\frac{t}{2}\right)\in\mathscr{L}_2(-\infty,\ \infty)\quad (k=1,\ 2,\ 3,\ 4). \tag{7}$$

As is well known [1, 2], the solution can be represented in the form of sums of the thermal potentials of a single and a double layer:

Scientific-Research Institute of Structural Physics, State Committee on Construction Activities of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 50, No. 4, pp. 654-659, April, 1986. Original article submitted February 7, 1985.

$$u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_1(s)}{\sqrt{t-s}} \exp\left[-\frac{(x-l_1-ps)^2}{4a^2(t-s)}\right] ds +$$

$$+ \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_2(s)(x-l_2-ps)}{(t-s)^{3/2}} \exp\left[-\frac{(x-l_2-ps)^2}{4a^2(t-s)}\right] ds, \quad (8)$$

$$v(y,\ t) = \frac{b}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_3(s)}{\sqrt{t-s}} \exp\left[-\frac{(y-l_3-qs)^2}{4b^2(t-s)}\right] ds +$$

$$+ \frac{1}{4b\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_4(s)(y-l_4-qs)}{(t-s)^{3/2}} \exp\left[-\frac{(y-l_4-qs)^2}{4b^2(t-s)}\right] ds, \quad (9)$$

where $\rho_i(t)$ $(i = 1, 2, 3, 4)$ are unknown functions determined from the boundary conditions (3)-(6); the initial conditions (2) are satisfied automatically. To determine the functions $\rho_i(t)$ $(i = 1, 2, 3, 4)$ from the boundary conditions (3)-(6), we obtain the system of integral equations

$$h_1(t) = -\frac{\alpha_{11}}{2}\rho_1(t) - \frac{\alpha_{13}}{2}\rho_2(t) + \sum_{k=1}^{4} c_{1k} \int_{-\infty}^{t} K_{1k}(t-s)\rho_k(s)\,ds, \quad (10)$$

$$h_2(t) = -\frac{\alpha_{21}}{2}\rho_1(t) + \frac{\alpha_{23}}{2}\rho_4(t) + \sum_{k=1}^{4} c_{2k} \int_{-\infty}^{t} K_{2k}(t-s)\rho_k(s)\,ds, \quad (11)$$

$$h_3(t) = \frac{\alpha_{32}}{2}\rho_2(t) - \frac{\alpha_{33}}{2}\rho_3(t) + \sum_{k=1}^{4} c_{3k} \int_{-\infty}^{t} K_{3k}(t-s)\rho_k(s)\,ds, \quad (12)$$

$$h_4(t) = \frac{\alpha_{42}}{2}\rho_2(t) + \frac{\alpha_{44}}{2}\rho_4(t) + \sum_{k=1}^{4} c_{4k} \int_{-\infty}^{t} K_{4k}(t-s)\rho_k(s)\,ds, \quad (13)$$

where

$$c_{m1} = \frac{2\alpha_{m2}a^2 - \alpha_{m1}p}{4a\sqrt{\pi}} \quad (m = 1,\ 2);$$

$$c_{12} = c_{22} = c_{31} = c_{32} = c_{41} = c_{42} = \frac{1}{4a\sqrt{\pi}};$$

$$c_{14} = c_{23} = c_{24} = c_{34} = c_{43} = c_{44} = \frac{1}{4b\sqrt{\pi}};$$

$$c_{13} = \frac{2\alpha_{14}b^2 - \alpha_{13}q}{4b\sqrt{\pi}}; \quad c_{33} = \frac{\alpha_{33}q + 2\alpha_{34}b^2}{4b\sqrt{\pi}},$$

with the kernels

$$K_{11}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{p^2x}{4a^2}\right);$$

$$K_{12}(x) = \left[-\frac{\alpha_{11}\mu^2}{2a^2x^{5/2}} + \frac{\mu(\alpha_{11}p - \alpha_{12}a^2)}{a^2x^{3/2}} + \frac{2pa^2\alpha_{12} - \alpha_{11}p^2}{2a^2x^{1/2}}\right] \exp\left[-\frac{\mu^2}{4a^2x} + \frac{\mu p}{2a^2} - \frac{p^2x}{4a^2}\right];$$

$$K_{13}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{q^2x}{4b^2}\right);$$

$$K_{14}(x) = \left[-\frac{\alpha_{13}\nu^2}{2b^2x^{5/2}} + \frac{\nu(\alpha_{13}q - \alpha_{14}b^2) + \alpha_{13}b^2}{b^2x^{3/2}} + \right.$$

$$\left. + \frac{2qb^2\alpha_{14} - \alpha_{13}q^2}{2b^2x^{1/2}}\right] \exp\left[-\frac{\nu^2}{4b^2x} + \frac{\nu q}{2b^2} - \frac{q^2x}{4b^2}\right];$$

$$K_{21}(x)=K_{11}(x);$$

$$K_{22}(x)=\left[-\frac{\alpha_{21}\mu^2}{2a^2x^{5/2}}+\frac{\mu(\alpha_{21}p-\alpha_{22}a^2)+\alpha_{21}a^2}{a^2x^{3/2}}+\frac{2pa^2\alpha_{22}-\alpha_{21}p^2}{2a^2x^{1/2}}\right]\exp\left[-\frac{\mu^2}{4a^2x}+\frac{\mu p}{2a^2}-\frac{p^2x}{4a^2}\right];$$

$$K_{23}(x)=\left[-\frac{\alpha_{23}\nu}{x^{3/2}}+\frac{\alpha_{23}q+2\alpha_{24}b^2}{x^{1/2}}\right]\exp\left[-\frac{\nu}{4b^2x}-\frac{q^2x}{4b^2}\right];$$

$$K_{24}(x)=\left[\frac{2b^2q\alpha_{24}-\alpha_{23}q^2}{2b^2\sqrt{x}}+\frac{\alpha_{23}}{x^{3/2}}\right]\exp\left(-\frac{q^2x}{4b^2}\right);$$

$$K_{31}(x)=\left[\frac{-\alpha_{31}\mu}{x^{3/2}}+\frac{2\alpha_{32}a^2-\alpha_{31}p}{\sqrt{x}}\right]\exp\left[-\frac{\mu^2}{4a^2x}+\frac{\mu}{2a^2}-\frac{p^2x}{4a^2}\right];$$

$$K_{32}(x)=\left(\frac{\alpha_{31}}{x^{3/2}}+\frac{2a^2\alpha_{32}p-\alpha_{31}p^2}{2a^2\sqrt{x}}\right)\exp\left(-\frac{p^2x}{4a^2}\right);$$

$$K_{33}(x)=K_{13}(x);$$

$$K_{34}(x)=\left[\frac{\alpha_{33}\nu}{2b^2x^{5/2}}+\frac{\alpha_{33}}{2b^2}+\frac{2b^2\alpha_{33}-\alpha_{34}\nu}{2b^2x^{3/2}}+\right.$$
$$\left.+\frac{2b^2\alpha_{34}q-\alpha_{33}q^2}{2b^2\sqrt{x}}\right]\exp\left[-\frac{\nu^2}{4b^2x}+\frac{q\nu}{2b^2}-\frac{q^2x}{4b^2}\right];$$

$$K_{41}(x)=\left[-\frac{\alpha_{41}\mu}{x^{3/2}}+\frac{2\alpha_{41}a^2-\alpha_{41}p}{x^{1/2}}\right]\exp\left[-\frac{\mu^2}{4a^2x}-\frac{p\mu}{2a^2}-\frac{p^2x}{4a^2}\right];$$

$$K_{42}(x)=\left(\frac{\alpha_{41}}{x^{3/2}}+\frac{2a^2\alpha_{42}p-\alpha_{41}p^2}{2a^2x^{1/2}}\right)\exp\left(-\frac{p^2x}{4a^2}\right);$$

$$K_{43}(x)=\left(-\frac{\alpha_{43}\nu}{x^{3/2}}+\frac{2\alpha_{44}b^2-\alpha_{43}q^2}{\sqrt{x}}\right)\exp\left(-\frac{\nu^2}{4b^2x}-\frac{q\nu}{2b^2}-\frac{q^2x}{4b^2}\right);$$

$$K_{44}(x)=\left(\frac{\alpha_{43}}{x^{3/2}}+\frac{2b^2\alpha_{44}q-\alpha_{43}q^2}{2b^2\sqrt{x}}\right)\exp\left(-\frac{q^2x}{4b^2}\right);$$

$$l_2-l_1=\mu;\quad l_4-l_3=\nu.$$

We convert this system of integral equations (10)-(13) into a system whose kernels depend on the product of the arguments. For this we introduce new variables through the formulas $t = \ln\tau$ and $s = -\ln\sigma$ and the new unknown functions $\varphi_k(\tau) = \rho_k(-\ln\tau)/\tau$ (k = 1, 2, 3, 4). We designate

$$g_k(\tau)=2h_k(\ln\tau)\quad(k=1,\ 2,\ 3,\ 4),$$

$$\tilde{K}_{ij}(x)=K_{ij}(\ln x)\quad(i,\ j=1,\ 2,\ 3,\ 4).$$

The functions $\tilde{K}_{ij}(x)$ are the kernels of the transformed system of integral equations. Let us examine the mean values of the functions $\tilde{K}_{ij}(x)$ (i, j = 1, 2, 3, 4) in the segment [1, x]:

$$R_{ij}(x)=\frac{1}{x}\int_1^x\tilde{K}_{ij}(x)\,dx\quad(i,\ j=1,\ 2,\ 3,\ 4).$$

The functions $R_{ij}(x)$ (i = 1, 2, 3, 4) belong to $\mathscr{L}_2(1, \infty)$ under the conditions

$$\frac{p^2}{4a^2}>\frac{1}{2},\quad\frac{q^2}{4b^2}>\frac{1}{2}.$$

We shall assume that these conditions are met, and hence we apply the method of solving systems of integral equations based on the expansion of integral operators with respect to orthogonal Watson operators [3-5]. We expand $R_{ij}(x)$ (i, j = 1, 2, 3, 4) in the interval $1 \leq x < \infty$ with respect to the functions $x^{-1}L_n(\ln x)$ (n = 0, 1, 2,...), forming a complete orthonormal system in $\mathscr{L}_2(1, \infty)$:

$$R_{ij}(x)=\sum_{n=0}^{\infty}(-1)^n a_{ijn}x^{-1}L_n(\ln x),$$

where $a_{ijn} = (-1)^n \int_1^\infty R_{ij}(x)x^{-1}L_n(\ln x)dx$. Here $L_n(\ln x)$ are Laguerre polynomials, defined through the equation

$$L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k n!\, z^k}{(n-k)!\,(k!)^2}.$$

We note that in the general case the functions $R_{ij}(x)$ cannot be expressed through elementary functions, and the coefficients a_{ijn} (i, j = 1, 2, 3, 4), n = 0, 1, 2, ... can be found by one of the numerical methods. After determining these coefficients, with allowance for the fact that

$$\int_{1/\tau}^{\infty} \tilde{K}_{ij}(\sigma\tau)\varphi_j(\sigma)\,d\sigma = \frac{d}{d\tau}\left\{\tau \int_{1/\tau}^{\infty} R_{ij}(\tau\sigma)\varphi_j(\sigma)\,d\sigma\right\},$$

and using the expansion of $R_{ij}(x)$ in functions $x^{-1}L_n(\ln x)$ (i, j = 1, 2, 3, 4), n = 0, 1, 2, ..., we obtain

$$\int_{1/\tau}^{\infty} \tilde{K}_{ij}(\sigma\tau)\varphi_j(\sigma)\,d\sigma = \sum_{n=0}^{\infty} a_{ijn} \frac{d}{d\tau}\left\{\tau \int_{1/\tau}^{\infty} (-1)^n x^{-1} L_n(\ln\sigma\tau)\varphi_j(\sigma)\,d\sigma\right\}. \quad (14)$$

Taking into account the fact that the functions $x^{-1}L_n \ln(x)$ are kernels of the Watson operators $(-1)^n S(TS)^n$, n = 0, 1, 2, ..., from (14) we obtain

$$\int_{1/\tau}^{\infty} \tilde{K}_{ij}(\sigma\tau)\varphi_j(\sigma)\,d\sigma = \sum_{n=0}^{\infty} a_{ijn} S(TS)^n \varphi_j(\tau).$$

Now we can represent the system of integral equations (10)-(13) in the following operator form

$$g_1(\tau) = -\alpha_{11}S\varphi_1(\tau) - \alpha_{13}S\varphi_2(\tau) + 2\sum_{k=1}^{4} c_{1k}\left[\sum_{n=0}^{\infty} a_{1kn}S(TS)^n\right]\varphi_k(\tau), \quad (15)$$

$$g_2(\tau) = -\alpha_{21}S\varphi_1(\tau) + \alpha_{23}S\varphi_4(\tau) + 2\sum_{k=1}^{4} c_{2k}\left[\sum_{n=0}^{\infty} a_{2kn}S(TS)^n\right]\varphi_k(\tau), \quad (16)$$

$$g_3(\tau) = \alpha_{32}S\varphi_2(\tau) - \alpha_{33}S\varphi_3(\tau) + 2\sum_{k=1}^{4} c_{3k}\left[\sum_{n=0}^{\infty} a_{3kn}S(TS)^n\right]\varphi_k(\tau), \quad (17)$$

$$g_4(\tau) = \alpha_{42}S\varphi_2(\tau) + \alpha_{44}S\varphi_4(\tau) + 2\sum_{k=1}^{4} c_{4k}\left[\sum_{n=0}^{\infty} a_{4kn}S(TS)^n\right]\varphi_k(\tau). \quad (18)$$

Applying the operator S to both sides of Eqs. (15)-(18), we obtain

$$Sg_1(\tau) = -\alpha_{11}E\varphi_1(\tau) - \alpha_{13}E\varphi_2(\tau) + 2\sum_{k=1}^{4} c_{1k}\left[\sum_{n=0}^{\infty} a_{1kn}(TS)^n\right]\varphi_k(\tau), \quad (19)$$

$$Sg_2(\tau) = -\alpha_{21}E\varphi_1(\tau) - \alpha_{23}E\varphi_4(\tau) + 2\sum_{k=1}^{4} c_{2k}\left[\sum_{n=0}^{\infty} a_{2kn}(TS)^n\right]\varphi_k(\tau), \quad (20)$$

$$Sg_3(\tau) = \alpha_{32}E\varphi_2(\tau) - \alpha_{33}E\varphi_3(\tau) + 2\sum_{k=1}^{4} c_{3k}\left[\sum_{n=0}^{\infty} a_{3kn}(TS)^n\right]\varphi_k(\tau), \quad (21)$$

$$Sg_4(\tau) = \alpha_{42}E\varphi_2(\tau) + \alpha_{44}E\varphi_4(\tau) + 2\sum_{k=1}^{4} c_{4k}\left[\sum_{n=0}^{\infty} a_{4kn}(TS)^n\right]\varphi_k(\tau), \quad (22)$$

where E is the identity operator.

Designating the operator determinant of this system (19)-(22) by the symbol Δ and the operators, which formally are algebraic augmented matrices of the coefficients to the unknowns, by the symbols A_{ij} (i, j = 1, 2, 3, 4), we find the solution of the system in the form

$$\varphi_k(\tau) = \Delta^{-1} \sum_{i=1}^{4} A_{ki} S g_i(\tau) \quad (k = 1,\ 2,\ 3,\ 4)$$

or

$$\varphi_k(\tau) = 2\Delta^{-1} \sum_{i=1}^{4} A_{ki} S h_i(\ln \tau) \quad (k = 1,\ 2,\ 3,\ 4).$$

The unknown functions $\rho_i(t)$ $(i = 1, 2, 3, 4)$ can now be expressed by the equations

$$\rho_k(t) = \exp(-t)\,\varphi_k[\exp(-t)] \quad (k = 1,\ 2,\ 3,\ 4).$$

Substituting these intensities into Eqs. (8)-(9) for the thermal potentials of a single and a double layer, we obtain the solution of the original problem in the form

$$u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\exp(-\tau)\,\varphi_1[\exp(-\tau)]}{\sqrt{t-\tau}} \exp\left[-\frac{(x-l_1-p\tau)^2}{4a^2(t-\tau)}\right] d\tau +$$

$$+ \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \frac{\exp(-\tau)\,\varphi_2[\exp(-\tau)](x-l_2-p\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{(x-l_2-p\tau)^2}{4a^2(t-\tau)}\right] d\tau,$$

$$v(y,\ t) = \frac{b}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\exp(-\tau)\,\varphi_3[\exp(-\tau)]}{\sqrt{t-\tau}} \exp\left[-\frac{(y-l_3-q\tau)^2}{4b^2(t-\tau)}\right] d\tau +$$

$$+ \frac{1}{4b\sqrt{\pi}} \int_{-\infty}^{t} \frac{\exp(-\tau)\,\varphi_4[\exp(-\tau)]}{(t-\tau)^{3/2}} \exp\left[-\frac{(-y-l_4-q\tau)^2}{4b^2(t-\tau)}\right] d\tau.$$

NOTATION

u, v, temperatures of the first and second linear heat conductors; x, y, coordinates; t, time; a^2, b^2, coefficients of thermal diffusivity of the first and second heat conductors; α_{ik} (i, k = 1, 2, 3, 4), constant coefficients in the boundary conditions of the system; $h_k(t)$ (k = 1, 2, 3, 4), assigned functions in the boundary conditions of the system; p, q, ℓ_k (k = 1, 2, 3, 4), constants in the law of motion of the boundary.

LITERATURE CITED

1. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
2. G. Myunts, Integral Equations [in Russian], Vol. 1, Gos. Izd. Tekhnikoteor. Lit., Moscow (1934).
3. M. A. Bartoshevich, "On one heat-conduction problem," Inzh.-Fiz. Zh., 28, No. 2, 340-345 (1975).
4. M. A. Bartoshevich, "Application of expansions in one orthogonal system of Watson operators for the solution of heat-conduction problems," Inzh.-Fiz. Zh., 28, No. 3, 516-522 (1975).
5. M. A. Bartoshevich and A. P. Prudnikov, "Application of Watson operators to the solution of certain problems of the theory of heat conduction," Inzh.-Fiz. Zh., 37, No. 3, 503-507 (1979).